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Table showing the Value of a Life Interest—continued.

| Premium<br>to insure £100. | Value of<br>Life Interest. | Δ. | Premium<br>to insure £100. | Value of<br>Life Interest. | Δ. |
|----------------------------|----------------------------|----|----------------------------|----------------------------|----|
| £ s. d.                    |                            |    | £ s. d.                    |                            |    |
| 5 1 0                      | 9·192                      | 52 | 6 11 0                     | 7·840                      | 39 |
| 5 2 0                      | 9·140                      | 51 | 6 12 0                     | 7·801                      | 38 |
| 5 3 0                      | 9·089                      | 51 | 6 13 0                     | 7·763                      | 38 |
| 5 4 0                      | 9·038                      | 50 | 6 14 0                     | 7·725                      | 38 |
| 5 5 0                      | 8·988                      | 50 | 6 15 0                     | 7·687                      | 38 |
| 5 6 0                      | 8·938                      | 49 | 6 16 0                     | 7·649                      | 37 |
| 5 7 0                      | 8·889                      | 48 | 6 17 0                     | 7·612                      | 37 |
| 5 8 0                      | 8·841                      | 49 | 6 18 0                     | 7·575                      | 37 |
| 5 9 0                      | 8·792                      | 47 | 6 19 0                     | 7·538                      | 36 |
| 5 10 0                     | 8·745                      | 47 | 7 0 0                      | 7·502                      | 36 |
| 5 11 0                     | 8·698                      | 47 | 7 1 0                      | 7·466                      | 36 |
| 5 12 0                     | 8·651                      | 47 | 7 2 0                      | 7·430                      | 35 |
| 5 13 0                     | 8·604                      | 46 | 7 3 0                      | 7·395                      | 35 |
| 5 14 0                     | 8·558                      | 45 | 7 4 0                      | 7·360                      | 35 |
| 5 15 0                     | 8·513                      | 45 | 7 5 0                      | 7·325                      | 34 |
| 5 16 0                     | 8·468                      | 45 | 7 6 0                      | 7·291                      | 35 |
| 5 17 0                     | 8·423                      | 44 | 7 7 0                      | 7·256                      | 34 |
| 5 18 0                     | 8·379                      | 44 | 7 8 0                      | 7·222                      | 33 |
| 5 19 0                     | 8·335                      | 43 | 7 9 0                      | 7·189                      | 34 |
| 6 0 0                      | 8·292                      | 43 | 7 10 0                     | 7·155                      | 33 |
| 6 1 0                      | 8·249                      | 43 | 7 11 0                     | 7·122                      | 33 |
| 6 2 0                      | 8·206                      | 42 | 7 12 0                     | 7·089                      | 32 |
| 6 3 0                      | 8·164                      | 41 | 7 13 0                     | 7·057                      | 33 |
| 6 4 0                      | 8·123                      | 42 | 7 14 0                     | 7·024                      | 32 |
| 6 5 0                      | 8·081                      | 41 | 7 15 0                     | 6·992                      | 31 |
| 6 6 0                      | 8·040                      | 41 | 7 16 0                     | 6·961                      | 32 |
| 6 7 0                      | 7·999                      | 40 | 7 17 0                     | 6·929                      | 31 |
| 6 8 0                      | 7·959                      | 40 | 7 18 0                     | 6·898                      | 31 |
| 6 9 0                      | 7·919                      | 40 | 7 19 0                     | 6·867                      | 31 |
| 6 10 0                     | 7·879                      | 39 | 8 0 0                      | 6·836                      | —  |

*On the Calculus of Finite Differences, and its Application to Problems in the Doctrine of Compound Interest and Certain Annuities. By WILLIAM CURTIS OTTER, F.R.A.S.\**

(Continued from p. 340, vol. vii.)

THE idea of periodic or discontinuous functions was primitively introduced by Euler, and has since been the subject of extended investigation by M. Fourier, who has made some new and important applications of it in his mathematical theory of heat.

\* Our attention has been called, by Mr. Benjamin F. Stevens, Secretary of the New England Mutual Life Insurance Company, at Boston, U.S., to the fact that the substance

It is the nature of those periodic or discontinuous functions, to preserve the same value for an infinite series of values of the corresponding variables subjected to a certain law, and which must be necessarily added to the integrals of the equations of finite differences in order to render them sufficiently general, as simple arbitrary constants are added to all quadratures in order to complete their generality.

Having thus briefly described the nature of periodic functions, I shall now proceed to demonstrate, by a plan similar to that adopted by J. F. W. Herschell, A.M., &c., which is the best that has come under my notice, some of the most important theorems in connection with the subject under consideration.

1. *To find an analytical function of  $x$ , such that, by giving to  $x$  all integer values from 0 to infinity, it shall give, in regular periodic rotation, the  $n$  values,  $a, b, c \dots k, a, b, c \dots k$ , &c.*

Let  $S_x$  represent a function of  $x$  which  $=1$ , when  $x=0, n$ , or a multiple of  $n$ , but which vanishes in every other case. Then, if  $\alpha, \beta, \gamma \dots \nu$ , be the  $n$ th roots of unity,  $S_x = \frac{\alpha^x + \beta^x + \gamma^x + \dots \nu^x}{n}$ ; hence it is evident that the function required is

$$P_x = a.S_x + b.S_{x-1} + \dots k.S_{x-n+1}.$$

Now, some one of the values  $x, x-1, x-2 \dots x-n+1$ , must necessarily be such a multiple; and  $x$  being made to vary from zero to infinity, such multiple will be either  $x, x-1, x-2 \dots$  or  $x-n+1$ ; in periodic order, therefore, the function  $P_x$  will become  $a, b, c \dots k$  in the same succession, and which is, therefore, the function sought.

*Obs.*—The roots of unity on which the above method is based are of great use in analysis, and possess many remarkable properties, as may be seen on reference to works on the theory of equations. The method by which they are obtained rests entirely on this—that the exponential  $a^x$  undergoes the extraction of the  $n$ th root by substituting  $\frac{x}{n}$  in lieu of  $n$ ; that every integral value of  $m$  makes  $\cos. 2m\pi + \sqrt{-1} \sin. 2m\pi = 1$ ; that this expression is of the form  $a^x$ , being  $\epsilon^{2m\pi\sqrt{-1}}$ ; and that, consequently, one of the  $n$ th

of the introductory portion of the paper of which this is a continuation has already appeared in a translation of M. Auguste Comte's work, entitled *Cours de Philosophie Positive*, made by Professor Gillespie, of Union College, in the State of New York. We have, accordingly, pointed this out to Mr. Otter, and that gentleman states that it was fully his intention to have made the acknowledgment due under such circumstances, and that his not having done it was accidental.—ED. A. M.

roots of unity is obtained by writing  $\frac{2m\pi}{n}$  for  $2m\pi$  in that expression.

Every integral power of an  $n$ th root is also an  $n$ th root; for, let  $a^n=1$ , then  $(a^m)^n=(a^n)^m=(1)^m=1$ , or  $a^m$  is an  $n$ th root of unity; therefore,  $a$  being one root,  $a^2, a^3, a^4 \dots (a^n=1)$  are roots also.

2. To find a function,  $P_x$ , which shall give the same values, in regular periodic succession, as those of  $n$  other given functions,  $a_x, b_x \dots k_x$ .

Since, by theorem 1,

$$P_x = a \cdot S_x + b \cdot S_{x-1} + \dots k \cdot S_{x-n+1}$$

generates the values  $a, b, c \dots k$ , by taking  $x$  successively  $= 0, 1, 2 \dots (n-1), (n), (n+1)$ , it follows that the function that generates the values  $\frac{a_x}{n}, \frac{b_x}{n} \dots \frac{k_x}{n}$ , by giving  $x$  the successive values  $0, 1, 2 \dots n, n+1$ , must be

$$P_x = a_x \cdot S_x + b_x \cdot S_{x-1} + c_x \cdot S_{x-2} + \dots k_x \cdot S_{x-n+1}.$$

*Def.*—These functions are denominated *circulating functions* of the  $n$ th order, and may be composed of constant or variable coefficients, as above instanced.

From the above equation, it is evident that

$$f(P_x) = f(a_x \cdot S_x) + f(b_x \cdot S_{x-1}) + f(c_x \cdot S_{x-2}) + \dots f(k_x \cdot S_{x-n+1});$$

*i.e.*, any function of a circulating function is a circulating function whose coefficients are similar functions of those of the original ones respectively.

Also, if  $Q_x, R_x$ , &c., represent other circulating functions, we have

$$f(P_x \cdot Q_x \cdot R_x \dots) = f(a_x \cdot a'_x \dots) S_x + f(b_x \cdot b'_x \dots) S_{x-1} + \dots f(k_x \cdot k'_x \dots) S_{x-n+1}.$$

For instance, if the coefficients be constant, we have

$$\frac{P_x + P'_x}{1 - P_x P'_x} = \frac{a + a'}{1 - aa'} S_x + \frac{b + b'}{1 - bb'} S_{x-1} + \dots \&c.$$

*Obs.*—Care must be taken not to generalize these equations, for  $f(P_x \cdot Q_x \dots)$  is itself a circulating function which passes through the cycle of values,

$$f(a_x \cdot a'_x \dots), f(b_x \cdot b'_x \dots) \dots \&c.$$

3. Circulating functions may be doubled, trebled, &c., in order, by assuming new circulating functions with doubled, trebled, &c., cycles of values. *Ex. gr.*:

$$\begin{aligned} 3_x \cdot a + 3_{x-1} \cdot b + 3_{x-2} \cdot c = \\ 6_x \cdot a + 6_{x-1} \cdot b + 6_{x-2} \cdot c + 6_{x-3} \cdot a + 6_{x-4} \cdot b + 6_{x-5} \cdot c. \end{aligned}$$

This is, I think, too evident to require a formal demonstration.

4. *Any symmetrical function of  $S_x, S_{x-1} \dots S_{x-n+1}$  is invariable*:—for, when  $x$  varies from 0 to  $\infty$ , some one of the values of these expressions must be *unity* and the other zero; and, as the function is symmetrical, it is immaterial in what order these values occur. The function has, therefore, the same value whichever of its elements become unity, the rest being *all* zero; *i.e.*,

$$\begin{aligned} S_x + S_{x-1} + S_{x-2} + \dots S_{x-n+1} &= 1. \\ S_x \cdot S_{x-1} \cdot S_{x-2} \dots S_{x-n+1} &= 0. \end{aligned}$$

5. *In like manner every symmetrical function of the circulating functions  $P_x, P_{x-1}, \dots P_{x-n+1}$  is invariable, provided the coefficients of  $P_x, P_{x-1}, \&c.$ , be constant*:—for every such function is a symmetrical function of  $S_x, S_{x-1} \dots S_{x-n+1}$ , as will be seen on considering the properties of the roots of unity demonstrated in books on the theory of equations, by reason of which we have  $S_{x-n} = S_x, S_{x-n-1} = S_{x-1}, \&c.$

$$\begin{aligned} \therefore P_x &= a \cdot S_x + b \cdot S_{x-1} + \dots \dots \dots k \cdot S_{x-n+1}, \\ P_{x-1} &= k \cdot S_x + a \cdot S_{x-1} + b \cdot S_{x-2} \dots j \cdot S_{x-n+1}, \\ P_{x-2} &= j \cdot S_x + k \cdot S_{x-1} + a \cdot S_{x-2} \dots i \cdot S_{x-n+1}, \\ &\vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \\ P_{x-n+1} &= b \cdot S_x + c \cdot S_{x-1} + \dots \dots \dots a \cdot S_{x-n+1}. \end{aligned}$$

Hence, it is evident that any symmetrical function of the right-hand members of these equations will involve  $S_x, S_{x-1} \dots S_{x-n+1}$  symmetrically, and will, consequently, be invariable. Its value will also be equal to that of a function whose coefficients are  $a, b, c, \&c.$  For instance, if

$$\begin{aligned} P_x &= a \cdot S_x + b \cdot S_{x-1}, \text{ where } n=2, \\ P_{x-1} &= a \cdot S_{x-1} + b \cdot S_x; \\ \therefore P_x \cdot P_{x-1} &= (a \cdot S_x + b \cdot S_{x-1}) (a \cdot S_{x-1} + b \cdot S_x) \\ &= ab \cdot S_x^2 + ab \cdot S_{x-1}^2 = ab \cdot (S_x^2 + S_{x-1}^2) = ab, \end{aligned}$$

an *invariable* quantity; since, by theorem 4,

$$\begin{aligned} S_x + S_{x-1} + \dots S_{x-n+1} &= 1, \\ \text{and } S_x \cdot S_{x-1} \dots S_{x-n+1} &= 0; \end{aligned}$$

and, generally,

$$\begin{aligned} P_x \cdot P_{x-1} \cdot P_{x-2} \dots P_{x-n+1} &= a \cdot b \cdot c \dots k; \\ P_x + P_{x-1} + P_{x-2} + \dots P_{x-n+1} &= a + b + c + \dots k. \end{aligned}$$

*Obs.*—I have before stated that circulating functions may be doubled, trebled, &c., by assuming new circulating functions with

doubled, trebled, &c., cycles of values; hence we have an easy method of reducing two or more circulating functions, with different periods of circulation, to a common period. *Ex. gr.:* If

$$P_x = a.S_x + b.S_{x-1}, \text{ where } n=2,$$

$$P'_x = a.S_x + \beta.S_{x-1} + \gamma.S_{x-2}, \text{ where } n=3,$$

we have, by trebling the first and doubling the second,

$$P_x = a.S_x + b.S_{x-1} + a.S_{x-2} + b.S_{x-3} + a.S_{x-4} + b.S_{x-5},$$

$$P'_x = a.S_x + \beta.S_{x-1} + \gamma.S_{x-2} + a.S_{x-3} + \beta.S_{x-4} + \gamma.S_{x-5},$$

which are both reduced to the common period of *six*.

The reason of the above process is obvious, for, when either  $x$ ,  $x-2$ , or  $x-4$ , is a multiple of 6, it is also a multiple of 2,  $\therefore$  both functions reduce themselves to  $a$ ; also, if  $x-1$ ,  $x-3$ , or  $x-5$ , be a multiple of 6, and consequently of 2,  $x-1$  must necessarily be a multiple of 2, in which case they both reduce themselves to  $b$ ; and, generally, it may be shown that if

$$P_x = a.S_x + b.S_{x-1} + \dots k.S_{x-n+1}$$

$${}_1P_x = a_1.S_x + b_1.S_{x-1} + \dots k_1.S_{x-n_1+1}$$

$$\vdots$$

$${}_pP_x = a_p.S_x + b_p.S_{x-1} + \dots k_p.S_{x-n_p+1},$$

we have

$$P_x = (a.S_x + b.S_{x-1} + \dots k.S_{x-n+1}) + (a.S_{x-n} + b.S_{x-n-1} + \dots k.S_{x-2n+1}) + \&c.,$$

to  $n_1.n_2.n_3 \dots n_p$  terms,

$${}_1P_x = (a_1.S_x + b_1.S_{x-1} + \dots k_1.S_{x-n_1+1}) + (a_1.S_{x-n} + b_1.S_{x-n-1} + \dots k_1.S_{x-2n_1+1}) + \&c.,$$

to  $n.n_2.n_3 \dots n_p$  terms,

$$\vdots$$

$${}_pP_x = (a_p.S_x + b_p.S_{x-1} + \dots k_p.S_{x-n_p+1}) + (a_p.S_{x-n} + b_p.S_{x-n-1} + \dots k_p.S_{x-2n_p+1}) + \&c.,$$

to  $n.n_1.n_2 \dots n_{p-1}$  terms;

by which means the equations are all reduced to the common period of  $n.n_1.n_2 \dots n_p$ .

Hence the following general rule for the reduction of circulating equations with different periods to a common period:—

*Find the common measure of the separate periods, should they have one; then the continued product of all the periods, divided by this common measure, will give the common period sought.*

From this it is obvious that, should equations occur involving circulating functions with different periods of circulation, they may be integrated, some instances of which will be given hereafter. It would be out of place to give them here, as I must first say something about circulating equations, the nature of which is such that

their coefficients are circulating functions, and to resolve them it is clear that they must be reduced to equations of the ordinary kind, and one of the most important objects of the preceding theorems is to show how this may be best effected.

The following is a simple instance of a circulating equation—

$$u_{x+2} \pm P_x \cdot u_{x+1} \pm u_x = 0,$$

in which  $P_x$  is a circulating function of the second degree, or in which  $n=2$  and  $P_x = a.S_x + \beta.S_{x-1}$ .

Now there are various ways of solving such an equation as this, but the following, by J. F. W. Herschell, A.M., &c., is one of the most simple:—

$$\text{Let } u_x = v_x \sqrt{P_x},$$

$$\text{then } u_{x+2} = v_{x+2} \sqrt{P_{x+2}} = v_{x+2} \sqrt{P_x},$$

by theorem 5; hence, by substitution, we have

$$v_{x+2} \sqrt{P_x} \pm P_x \cdot v_{x+1} \sqrt{P_{x+1}} \pm v_x \sqrt{P_x} = 0;$$

$$\text{or, } v_{x+2} \pm v_{x+1} \sqrt{P_x \cdot P_{x-1}} \pm v_x = 0;$$

but, by theorem 5, the functional product  $P_x \cdot P_{x-1}$  is *invariable* and  $= a.b$ ;  $\therefore$  our equation becomes

$$v_{x+2} \pm \sqrt{ab} \cdot v_{x+1} \pm v_x = 0,$$

being an ordinary equation with constant coefficients, and easily integrated by the usual methods.

*Obs.*—In the next Number of the *Journal*, I hope to give a more general process of solving such equations as these, by assuming for the independent variables a circulating function with unknown and variable coefficients, which will be found to apply to all kinds of circulating equations, together with a method of integrating equations involving circulating functions with different periods of circulation, with the application of the preceding theorems to some useful and interesting questions in the doctrine of compound interest annuities, continued fractions, &c., &c.

#### *Forms of Endorsement on Policies used in the Practice of Life Assurance.*

AS the forms above referred to are now numerous, and in constant requisition, we have thought that, by collecting some of them together and putting them into print, we might be affording certain facilities for, and leading the way to, more uniformity of practice in this particular than has hitherto been attainable. We, accord-